

Search for long-living topological solutions of nonlinear field theory φ^4

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Abstract

We look for long-living topological solutions of classical nonlinear $(1+1)$ -dimensional φ^4 field theory. As for that the original method “cut and match” is offered. In the framework of this method new long-living states are obtained in both topological sectors. In particular, a highly excited state of a kink is found in one case. We discover several ways of energy reset. In addition to the expected emission wave packets (with small amplitude) in the case of some selected initial conditions a large energy reset becomes a result of the production of kink-antikink pairs. Besides a topological number of a kink in the central region is changing in the contrast of saving full topological number. At lower excitation energies there is a long-living excited vibrational state of the kink. This phenomenon is the final stage of all considered initial states. Over time this excited state of the kink is changing to linearized well-known solution — a discrete kinks excitation mode. The proposed method yields a qualitatively new way of describing the large-amplitude bion, which was detected earlier in the kink scattering processes in the non-topological sector.

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I. INTRODUCTION

We consider the $\lambda\varphi^4$ theory with a real scalar field $\varphi(t, x)$ in $(1+1)$ dimensions [1–3]. Its dynamics determined by the Lagrangian:

$$\mathcal{L} = \frac{1}{2} \left(\frac{\partial\varphi}{\partial t} \right)^2 - \frac{1}{2} \left(\frac{\partial\varphi}{\partial x} \right)^2 - U(\varphi), \quad (1)$$

where $U(\varphi)$ — is a potential, defining the self-interaction of the field in the considered model [1]:

$$U(\varphi) = \frac{\lambda}{4} \left(\frac{m^2}{\lambda} - \varphi^2 \right)^2. \quad (2)$$

The plot of (2) is shown at figure 1 (left panel). We take up a model with a non-negative potential having two minima, so all static solutions with finite energy splits into disjoint classes, so called topological sectors, according to their asymptotic behavior at extremely large x . Solutions with $\varphi(-\infty) \neq \varphi(+\infty)$ are called topological, while those with $\varphi(-\infty) = \varphi(+\infty)$ — nontopological. Both types of the solutions are of growing interest in physics. In particular, they arise in the questions of three- or two-dimensional domain walls. However, a one-dimensional case also is curious and it was considered in different works for diverse models [4–6]. In the $\lambda\varphi^4$ model there is a soliton solution called a kink, in [7], [8] a phenomenon of “wobbling kink” was studied. Moreover, a three- or two-dimensional domain wall presents a one-dimensional kink interpolating two different vacua of the model. In some cases for considering models the approximate methods are offered [9]. The domain walls in the $\lambda\varphi^4$ model can be applied in some cosmological models, for example, during discussions of the dark matter and dark energy themes [10]. The results of numerical simulations in other models [9] can be applied in the solid body physics [11].

The Lagrangian (1) with (2) yields the equation of motion for $\varphi(t, x)$ after transition to dimensionless variables:

$$\varphi_{tt} - \varphi_{xx} - \varphi + \varphi^3 = 0. \quad (3)$$

At the next step we find the analytical solutions of the equation (3) and research them. Note, that the vacua of this model $\varphi_{\text{vac}}^{(1)} = -1$ and $\varphi_{\text{vac}}^{(2)} = +1$ are stable solutions of (3). Moreover there is unstable permanent solution $\varphi = 0$ with infinite energy.

Except previous solutions there is also a static non-trivial topological solution like solitary

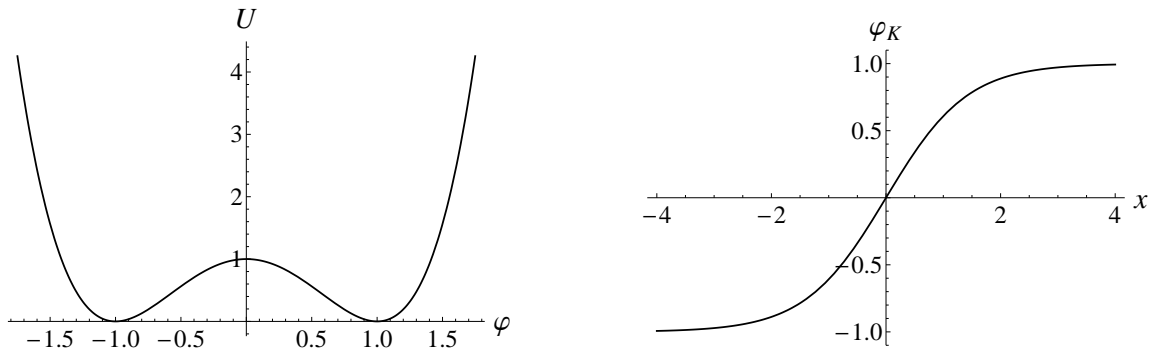


FIG. 1: The potential (2) of the $\lambda\varphi^4$ model (left panel) and the kink solution (4) (right panel).

wave called a kink [1]. It can be easily found by solving static equation (3):

$$K \equiv \varphi_K(x - x_0) = \tanh \frac{(x - x_0)}{\sqrt{2}}. \quad (4)$$

The antikink \bar{K} is given as minus K . An energy functional for the Lagrangian (1), in static case (4), is calling a mass of the kink $M_K = 2\sqrt{2}/3$. The plot of (4) is presented at figure 1 (right panel).

A. Excitation spectrum of the kink

In order to analyse the excitation spectrum of the static kink, we add to it a small perturbation $\delta\varphi$. In other words we make such substitution to (3) as

$$\varphi(t, x) = \varphi_K(x) + \delta\varphi(t, x) = \varphi_K(x) + e^{i\omega t}\psi(x). \quad (5)$$

Taking the terms linear in $\delta\varphi$ (5), we obtain the next equation:

$$\hat{H}\psi = E\psi, \quad \hat{H} = -\frac{d^2}{dx^2} - 3 \cosh^{-2} \frac{x}{\sqrt{2}}, \quad E = \omega^2 - 2. \quad (6)$$

The eigenvalue $\omega_0 = 0$ belongs to the discrete part of the excitation spectrum (6) [1], but also there is one vibrational excitation:

$$\delta\varphi_1 = \psi_1(x)e^{i\omega_1 t}, \quad \text{where} \quad \psi_1(x) = \left(\frac{3}{2\sqrt{2}}\right)^{1/2} \tanh \frac{x}{\sqrt{2}} \cosh^{-1} \frac{x}{\sqrt{2}}, \quad \omega_1 = \sqrt{\frac{3}{2}}. \quad (7)$$

B. Analytical solution, depending on x

The above solutions are not a full set of solutions in the φ^4 model. Let's consider a static wave solution with infinite energy. We consider the static equation (3)

$$\varphi_{xx} = -\varphi + \varphi^3. \quad (8)$$

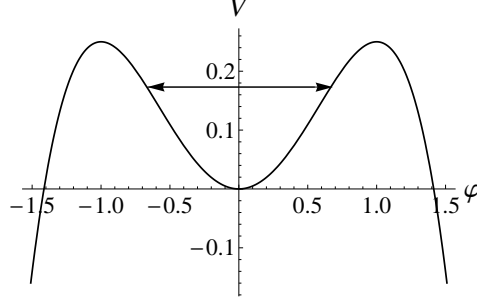


FIG. 2: The dimensionless potential (9). The arrow shows the domain of the solution $\varphi(x)$ for an arbitrarily chosen value $0 < \varphi_0 < 1$.

If one reduces the problem to Newton's equation, will obtain

$$\ddot{x} = F(x) = -\nabla V(x), \quad \text{where} \quad V(x \rightarrow \varphi) = \frac{\varphi^2}{2} - \frac{\varphi^4}{4} + \text{const.} \quad (9)$$

Figure 2 shows a plot of a $V(x \rightarrow \varphi)$. In this case $\varphi(x)$ describes a trajectory of an oscillatory movement between points $-\varphi_0$ and φ_0 . At $0 < \varphi_0 < 1$ oscillations are periodic. In the limiting case $\varphi_0 = 1$ the fluctuations disappear, because time to return to the starting point $\varphi_0 = 1$ reaches infinity.

A transition to a dimensionless variable $\varphi(x) = \varphi_0 \chi(x)$ and following notations

$$k^2 = \frac{\varphi_0^2/2}{1 - \varphi_0^2/2}, \quad b^2 = 1 - \frac{\varphi_0^2}{2},$$

where $0 \leq k^2 \leq 1$ and $1/2 \leq b^2 \leq 1$, leads us to:

$$\int_0^{\chi(x)} \frac{d\chi}{\sqrt{(1 - \chi^2)(1 - k^2\chi^2)}} = \langle \chi = \sin \psi \rangle = \int_0^{\arcsin \chi} \frac{d\psi}{\sqrt{(1 - k^2 \cos^2 \psi)}}.$$

The last integral — the elliptic integral of the first kind ($F(\arcsin \chi, k) = bx$) [12]. Then the static periodic solution of equation (8) can be written as

$$\varphi_{\text{el}}(x) = \varphi_0 \operatorname{sn}(bx, k), \quad (10)$$

where $\operatorname{sn}(bx, k)$ — the elliptic sine [12]. At small k (corresponding to $\varphi_0 \ll 1$) there is a concordance $\operatorname{sn}(z) \approx \sin(z)$. At $\varphi_0 \rightarrow 0$ the solution (10) becomes a permanent unstable solution $\varphi \sim 0$, previously noted. The plots of (10) are shown in the figure 3 for different values of parameter φ_0 . The elliptic sine period is calculated using the following formula [12]:

$$T = \frac{4F(\pi/2, k)}{\sqrt{1 - 0.5\varphi_0^2}}. \quad (11)$$

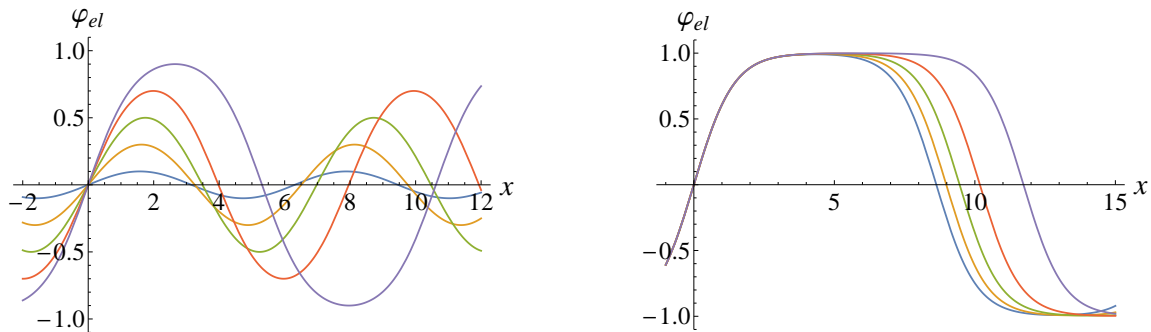


FIG. 3: The dependence of the solution $\varphi_{el}(x)$ on the parameter φ_0 . The plots are for different φ_0 from 0.1 to 0.9 with step 0.2 (left panel) and φ_0 from 0.991 to 0.999 with step 0.002 (right panel). The curves on the plot are in order the parameter φ_0 grows.

II. FORMULATION OF THE PROBLEM

The long-living solutions with high amplitude are of growing interest in classical field theory. This type of solutions, called bion or breather, was found early in the kink-antikink collisions in the φ^4 model both in one- [13–17] and three- dimensional cases [18].

We propose a method “cut and match” to find a long-living field configuration with using previously found solutions $\varphi = 0$, (4) and (10) of the equation (3). In this case a part of initial state is composed of the kink (4), which is divided for two equal pieces at $x = 0$. These halves of the kink are fixed at $\pm x_0$. Then, one of the solutions ($\varphi = 0$ or φ_{el} on the finite interval) of (3) is placed in the space between these two halves. An initial state, which is composed in described manner, is shown by figure 4.

Note, that if we take $\varphi = 0$, the initial state will become unstable. Its energy linearly increases with growing of the distance $2x_0$.

In other case we take a solution in terms of elliptic solution (10) for fixed value $0 < \varphi_0 < 1$. For a smooth screed of selected solutions one defines the value of x_0 as a half of the period T of the elliptic function φ_{el} . Thus we obtain an initial configuration $(-1, \varphi_0, 0, -\varphi_0, 1)$. In the following text the definition

$$(-1, \varphi_0, 0, -\varphi_0, 1)$$

means: in the area $-\infty < x < -T/2$ the initial state consists of a half of (4), in $-T/2 < x < +T/2$ — (10) (there $\varphi(x = -T/4) = \varphi_0$, $\varphi(x = 0) = 0$, $\varphi(x = T/4) = -\varphi_0$), and in the area $T/2 < x < \infty$ the solution consists of a second part of (4). There T is a period of elliptic function (11). The profile of this type of initial state is shown in the figure 4 (for

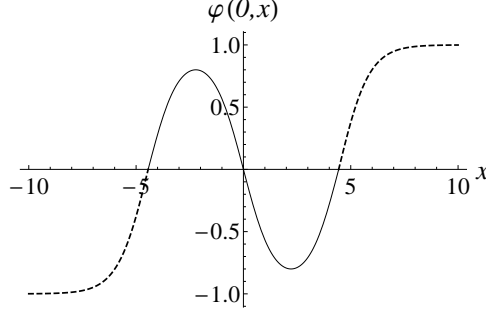


FIG. 4: The plot of the initial state, which is constructed with the method “cut and match”. A dashed line shows the half-kinks (4), a solid line shows a solution in terms of elliptic function φ_{el} for $\varphi_0 = 0.8$.

$\varphi_0 = 0.8$). Firstly, we consider a “static initial state” ($\partial_t \varphi_0 = 0$), but later we take into account some configurations with dynamics:

$$\frac{\partial \varphi_0(0, x)}{\partial t} = \frac{\varphi_0(\tau, x) - \varphi_0(0, x)}{\tau} = \delta \varphi_0 \neq 0.$$

Numerical solution of the equation of motion. We solve the partial differential equation (3) with convergent difference scheme with non-fixed boundary conditions, while derivatives are approximated by finite differences. The steps are taken as follows $h = 0.04$ (space step) and $\tau = 0.02$ (time step), the equation is solved from $t = 0$ to $t = 100$. This choice of steps helps to optimize a ratio accuracy of the obtained results and the duration of computing. During the evolution a check of law of conservation of energy is conducted, taking into account a flow of energy from fixed boundaries. The initial states are compiled with the use of the computer algebra system Mathematica 8.

A. Result for unstable vacuum $\varphi = 0$

First of all, the initial condition consists of two halves of the kink, placed in $\pm x_0$, and unstable zero solution $\varphi = 0$ between them. The energy increases linearly with growing value of x_0 . Two parts in the evolution are observed. Firstly, there is a convergence of both halves of kink with velocity equals the speed of light. When the halves finally meet each other, two processes alternate: a formation of loops and an emission of waves from the kink (so called, “wobbling kink”). The obtained solution φ_{sol} is close to linearized solution of equation (3), where $\varphi_{\text{sol}} \approx \varphi_K + \delta \varphi$, which lives very long and is characterized by small

emission of waves. These waves carry off some energy from the area of localization. Let's explain this phenomenon. At small values of $k(\varphi_0)$ the solution changes from $\varphi_{\text{el}} \approx \text{sn}x$ to $\varphi \approx \sin x$. As the sin is a periodic function, when $2x_0 \leq 2\pi$ the initial condition (with loops) comes not as an exciting of excited mode of elliptic function, but as an exciting of highly-amplitude vibration of a kink solution. So the evolution of the initial state can be described qualitatively by

$$\varphi_{\text{sol}} \approx \tanh\left(\frac{x}{\sqrt{2}}\right) \left(1 + \frac{A(t)}{\cosh(x/\sqrt{2})}\right), \quad A(t) = A_0 \cos \omega t. \quad (12)$$

The evolution can be described by (12) as there are two modes in the kinks spectrum. One of them, which correlates with (7), is responsible for small vibrations across the solution. There is an idea, that the observed vibrations stop being small, but still can be described with similar periodic function as $\cos \omega t$ (we take $\omega = \sqrt{3/2}$ like in (7)). In (12) the parameter A_0 is taken as constant. But in the numerical simulations it isn't a constant as there is a small emission from area of localisation of solution.

Moreover, there is one precondition to describe qualitatively an obtained solution precisely with (12). The function (12) equals zero in $x = 0$ one time, if $A_0 \geq -1$, and three times, if $A_0 < -1$. Note, that a quasi-periodic formation of the loops with period equals $\approx 2\pi$ — it is also one reason for using proposed phenomenological description. This period correlates with $\cos \omega t$:

$$T = \frac{2\pi}{\omega} \approx 2\pi, \text{ as } \omega = \sqrt{\frac{3}{2}} \approx 1.$$

In [7] is shown, that considering such substitution as $\varphi_K + \delta\varphi$ in the equation (3) in the quadratic approximation by $\delta\varphi$ gives us asymptotically stable solution. Its large amplitude vibrations are characterized by strong suppression. At figures 5 and 6 two parts of evolution and a comparison with analytical solution (12) are shown for two chosen moments of time.

At high values of x_0 the observed loops in the evolution are characterized by not small amplitudes. In this case the final states of evolution can be identify with the elliptic solution φ_{el} between two halves of kink.

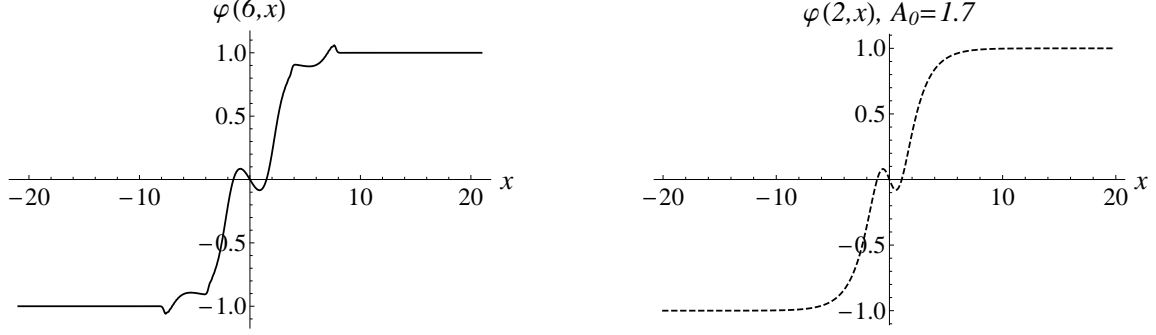


FIG. 5: The profile of $\varphi(t, x)$ at $t = 6$ (left panel) and mapped plot of the solution (12) for $A_0 = 1.7$ at $t = 2$ (right panel), $x_0 = 2$.

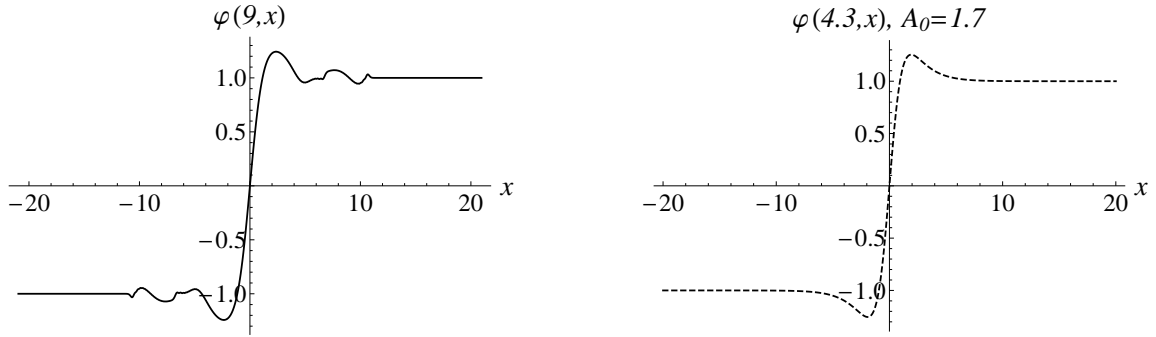


FIG. 6: The profile of $\varphi(t, x)$ at $t = 9$ (left panel) and mapped plot of the solution (12) for $A_0 = 1.7$ at $t = 4.3$ (right panel), $x_0 = 2$.

B. Result for an elliptic function with $0 < \varphi_0 < 1$

1. Dynamical initial state ($\delta\varphi_0 < 0$). Configuration $(-1, \varphi_0, -1)$.

In previous works a long-living configuration has been found, so called bion [1]. However, an analytical description of the observed process has not been given. In our work we take an initial condition $(-1, \varphi_0, -1)$, composed of one half-kink K and one half-antikink \bar{K} and a half of period of φ_{el} with fixed φ_0 . This initial state is dynamical ($\varphi_0 + \delta\varphi_0, \delta\varphi_0 < 0$). We obtain long-living state with oscillation of an amplitude of φ at $x = 0$ (figure 7). The observing oscillations in terms of an amplitude $\varphi_0(t)$ are called a regular bion. Also, it can be considered as a new form of description of early found bion [1].

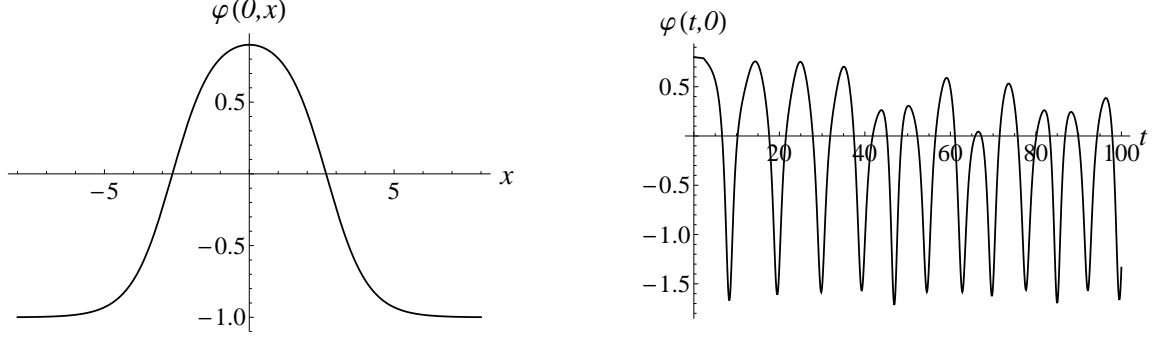


FIG. 7: The profile of $\varphi(t, x)$ at $t = 0$ (left panel) and the profile of $\varphi(t, 0)$ (right panel). Parameters: $\varphi_0 = 0.8$ and $\delta\varphi = -0.001$.

2. *Statistic initial state* ($\delta\varphi_0 = 0$). *Configuration* $(-1, \varphi_0, 0, -\varphi_0, 1)$.

We take an initial condition $(-1, \varphi_0, 0, -\varphi_0, 1)$ for $\varphi_0 > 0.7$ (for smoother stitching), while the observed evolution does not qualitatively depend on φ_0 . Further, for example, we show the results for the case $\varphi_0 = 0.8$.

We find two steps in the evolution: external (a loop of high amplitude formation) and internal (a highly deformed kink). After some time the loops continue their forming with smaller amplitude. After these 4 — 5 cycles the external step disappears and the solution starts to be presented by long-living excited kink with the wave packet emission from the area of localization. This phenomenon is called a “wobbling kink”. This state is a final step of the evolution, which is observed for other variants of initial states. The profiles of $\varphi(t, x)$ for $\varphi_0 = 0.8$ at some particular time are shown at figures 8 and 9.

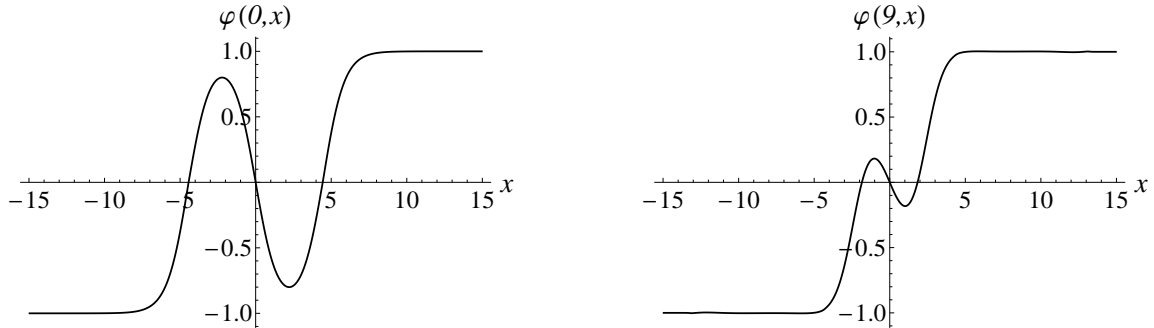


FIG. 8: The profiles of $\varphi(t, x)$ at $t = 0$ (left panel) and at $t = 9$ (right panel). Parameters: $\varphi_0 = 0.8$ and $\delta\varphi_0 = 0$.

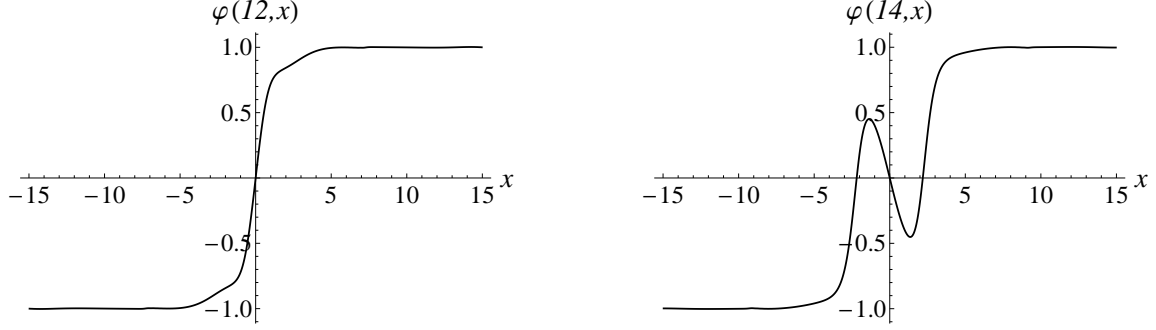


FIG. 9: The profiles of $\varphi(t, x)$ at $t = 12$ (left panel) and at $t = 14$ (right panel). Parameters: $\varphi_0 = 0.8$ and $\delta\varphi_0 = 0$.

3. *Dynamical initial state* ($\delta\varphi_0 < 0$). *Configuration* $(-1, \varphi_0, 0, -\varphi_0, 1)$.

An addition of $\delta\varphi_0 < 0$ to the initial state of the configuration $(-1, \varphi_0, 0, -\varphi_0, 1)$ leads to a faster reduction of the amplitude. At low values of $|\delta\varphi_0|$, $\delta\varphi_0 < 0$ the loops arise. For the first time during the evolution the kink-antikink pairs $K\bar{K}$ turn up. This phenomenon has a threshold. The increase of $|\delta\varphi_0|$ gives us a qualitatively new type of the evolution ($-0.0013 < \delta\varphi_0 < -0.0044$ for $\varphi_0 = 0.9$). In the system the next transition happens:

$$\mathbf{K} \rightarrow K\bar{K}K, \quad (13)$$

in the center of the configuration a topological number is changing. A transition (13) is shown at figure 10 for $\delta\varphi_0 = -0.0040$. The next increasing of $|\delta\varphi_0|$ ($-0.0045 \leq \delta\varphi_0 < \dots$ for $\varphi_0 = 0.9$) gives us the next transition

$$\mathbf{K} \rightarrow K\bar{K}K\bar{K}K. \quad (14)$$

Now we observe a saving of topological number in the center. These transitions are observed for different φ_0 . The transition (14) is shown at figure 11 for $\delta\varphi_0 = -0.0045$. We expect that with increasing the value of $|\delta\varphi_0|$ the similar qualitative changes will be observed in the initial conditions of the evolution of initial state.

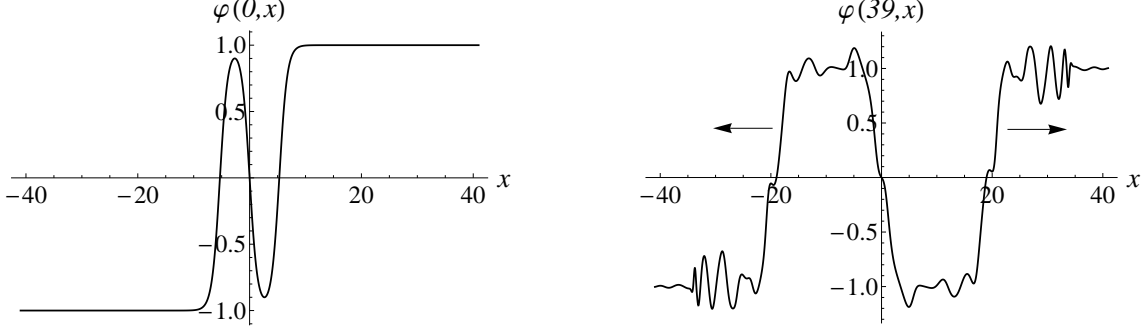


FIG. 10: The profiles of $\varphi(t, x)$ at $t = 0$ (left panel) and at $t = 39$ (right panel). The formation of (13). Parameters: $\varphi_0 = 0.9$ and $\delta\varphi_0 = -0.0040$. The arrows indicate the direction of movement of formed kinks.

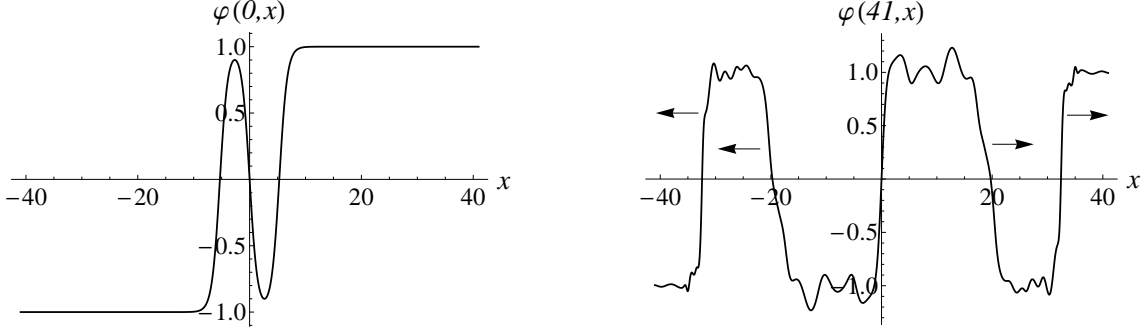


FIG. 11: The profiles of $\varphi(t, x)$ at $t = 0$ (left panel) and at $t = 41$ (right panel). The formation of (14). Parameters: $\varphi_0 = 0.9$ and $\delta\varphi_0 = -0.0045$. The arrows indicate the direction of movement of formed (anti)kinks.

Conclusion

We study the new long-living solutions in classical field theory model $\lambda\varphi^4$ in $(1+1)$ dimensions.

In this work we use the special method for forming of initial state for numerical simulations — the method “cut and match”. Using this method gives us new long-living solutions both for vacuum solutions and solutions with nontrivial topological number.

In previous work [13] a long-living configuration was observed in the kink-antikink scattering. It was called a bion. In current work the method “cut and match” gives us an opportunity to inject qualitative new way to describe a bion formation.

Furthermore, the highly excited states of the kink are observed in a sector with nontrivial topological number. We find a number of ways to reset this energy from this state. Except

for emission of wave packet with small amplitude, firstly, an arising of the kink-antikink pairs is observed. This phenomenon can be perceived as a way to energy reset. At the same time there is a change of the topological number of the kink, located in the central zone in the area. At lower excitation energies there is a long-living excited vibrational state of the kink. The phenomenon of a “wobbling kink” is final state all considered means of exciting. After some time this excited state of a kink turns to a linearized one, which was formerly known — a discrete mode of exciting kink.

Despite the large number of new results, the method “cut and match” has a wide range of outstanding issues in the application of the $\lambda\varphi^4$ model. In particular, more detailed study of the dynamic of the initial conditions for the case of $\delta\varphi_0 < 0$ will be interesting, because in the last case there is a phenomenon of the birth of new kink-antikink pairs.

In the conclusion, we would like to notice that this research, first of all, can be useful in the field theory. But, also it can be useful in other area of physics. The obtained results can be applied in elementary particle physics, cosmology, astrophysics and physics of condensed matter. With a high probability we expect that the obtained solutions can be implemented in the early stages of the evolution of the Universe.

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